8 / SPLIT-PLOT DESIGN— FACTORIAL DESIGN WITH BLOCK-TREATMENT CONFOUNDING

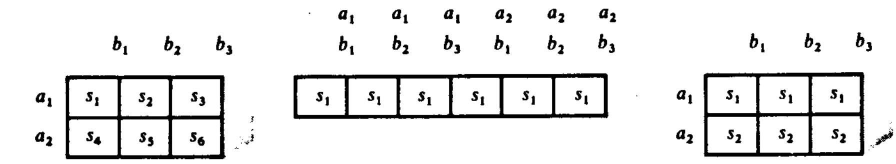
8.1 DESCRIPTION OF DESIGN

Subject heterogeneity is the rule rather than the exception in behavioral research. The randomized block design described earlier enables an experimenter to partially isolate the effect of subject heterogeneity in testing treatment effects. This is accomplished by using matched subjects or repeated measures on the same subject. In a randomized block design, blocks of subjects are composed in such a way that variation among subjects within each block is less than the variation among blocks. A split-plot design with repeated measures or matched subjects represents an extension of this principle to experiments having two or more treatments. This design is appropriate for experiments that meet, in addition to the general assumptions of the analysis of variance model, the following conditions:

1. Two or more treatments, with each treatment having two or more levels, that is, p levels of A, which is designated as a between-block or nonrepeated-measurements treatment, and q levels of B, which is designated as a within-block or repeated-measurements treatment, where p and $q \geq 2$.
2. The number of combinations of treatment levels is greater than the desired number of observations within each block.
3. If repeated measurements on the subjects are obtained, each block contains only one subject. If repeated measurements on the subjects are not obtained, each block contains q subjects.
4. For the repeated-measurements case, p samples of n subjects each from a population of subjects are randomly assigned to levels of the nonrepeated treatment (A). The sequence of administration of the repeated treatment levels in combination with one level of the nonrepeated treatment is randomized independently for each block. Exception to this procedure is made when the nature of the repeated treatment precludes randomization of the presentation order.
5. For the nonrepeated-measurements case, p samples of n blocks of q subjects from a population of subjects are randomly assigned to levels of treatment (A). After this, levels of treatment (B) are assigned randomly to the q subjects within each block.

A COMPARISON OF THREE FACTORIAL EXPERIMENTS

A graphic comparison of three experimental designs—a completely randomized factorial design, a randomized block factorial design, and a split-plot design—is shown in Figure 8.1-1. In this figure the split-plot



(*a*) Type CRF-23 design (*b*) Type RBF-23 design (*c*) Type SPF-2.3 design

Figure 8.1-1 Comparison of three types of designs.

repeated measures design is designated by SPF-2.3. The letters $s_1, s_2 \ldots s_6$ refer to sets of n subjects. In the type CRF-23 design, each set of subjects receives only one of the pq treatment combinations. An examination of part (a) reveals that the design is composed of two completely randomized designs. Subjects assigned to treatment level a_1 comprise one type CR-3 design, while subjects assigned to a_2 comprise the other type CR-3 design. The building block for the designs shown in parts (b) and (c) is a randomized block design. In a type RBF-23 design, a single set of subjects (s_1) receives all pq treatment level combinations. By contrast, in a split-plot design subjects in set s_1 receive only one level of treatment A but all levels of treatment B. The analysis of treatments A and B in a split-plot design, when viewed separately, resembles the analysis for a completely randomized design and a randomized block design, respectively. This analogy is discussed in Section 8.4.

SPECIAL FEATURES OF SPLIT-PLOT DESIGNS

Split-plot repeated measures designs in which a subject receives all levels of some treatments but only one level of other treatments are sometimes referred to as *mixed designs* (Lindquist, 1953). Winer (1962) uses the designation "multifactor experiments having repeated measures on some elements" for this class of designs.

The origin of the term *mixed design* as a designation for split-plot designs can be readily discerned from an inspection of Figure 8.1-1c. In this figure differences between levels a_1 and a_2 involve differences between s_1 and s_2 as well as the effects of treatment A. However, differences between any two levels of treatment B do not involve differences between

s_1 and s_2 because the same subjects are observed under all levels of B. In a type SPF-2.3 design, the main effects of treatment A are said to be completely confounded with differences between blocks or sets of subjects. The main effects of B and the interaction AB are free from such confounding. A confounding scheme in which a treatment is confounded with blocks does not affect the interpretability of the treatment effects, only the precision of the estimate. The effects of treatment A are described as *between-block (subject)* effects, while the effects of treatment B and interaction AB are described as *within-block (subject)* effects. Tests on B and AB are generally much more powerful than tests on A.

The general designation for a two-factor repeated-measures split-plot design is SPF-$p.q$. According to this designation all lower-case letters before the dot stand for the number of levels of between-block treatments; letters after the dot stand for levels of within-block treatments.

There are many research problems in the behavioral sciences where split-plot designs are especially appropriate. A general problem inherent in all behavioral research is subject heterogeneity. Differences among subjects are often such as to obscure treatment effects. A repeated measures or matched subjects design offers the advantage of controlling subject heterogeneity. In addition to this advantage, a repeated measures design is particularly useful in assessing certain types of treatment effects. For example, in experiments designed to investigate learning, transfer, fatigue, and so on, the use of repeated measures on the same subjects is often the simplest way to investigate the research problem. Randomization of the order-of-treatment level presentation for these kinds of variables is not always feasible, for the nature of the treatment dictates the order.

A LIMITATION OF THE USE OF REPEATED MEASURES

A word of caution concerning the use of repeated measurements on the same subject is in order. When matched subjects are assigned to within-block treatment levels, it may be assumed that estimates of treatment effects that have been obtained from the q cells are correlated. The model underlying type SPF-$p.q$ designs permits a particular kind of statistical dependency between observations in the q levels of B but requires that the error portion of these scores must be independent of each other and the treatment effects. There is ample reason to believe that in repeated measures experiments the error components of the scores are not independent and that the variance-covariance matrix departs from the required form. That is, the $q \times q$ repeated measures dispersion matrix does not have all diagonal elements equal to σ^2 and all off-diagonal elements equal to $\rho\sigma^2$. Procedures for investigating this issue are presented in Section 8.5. Bargmann (1957) presents a comprehensive discussion of homogeneity assumptions in repeated measures designs. Lana and Lubin

(1963) discuss in detail the problems resulting from unequal correlations among levels of the repeated treatment. Subsequent sections describe suggested procedures for coping with these problems.

The model underlying the F test for a split-plot repeated measures design does not include a term for sequence or carry-over effects. Thus repeated measurements on the same subject should be avoided for treatments in which the administration of one level affects performance on a subsequent level. An exception to this, of course, is when carry-over effects are the primary interest of the experimenter. Gaito (1961) has discussed the problem of *order effect* when repeated measures are obtained on the same subjects and has emphasized the importance of randomizing presentation of treatment levels.

8.2 LAYOUT AND COMPUTATIONAL PROCEDURES FOR TYPE SPF-*p.q* DESIGN

The layout of a type SPF-2.4 design is illustrated in Table 8.2-1. Let us assume that an experimenter is interested in vigilance performance. He has designed an experiment to evaluate the relative effectiveness of two modes of signal presentation during a four-hour monitoring period. Treatment A, which is designated as mode of signal presentation, has two levels, a_1 = auditory signal (tone) and a_2 = visual signal (light). Treatment B has four levels corresponding to successive monitoring periods: $b_1 = 1$ hour, $b_2 = 2$ hours, $b_3 = 3$ hours, and $b_4 = 4$ hours. The research hypotheses leading to this experiment can be evaluated by means of statistical tests of the following null hypotheses:

$$H_0: \alpha_i = 0 \quad \text{for all } i$$
$$H_1: \alpha_i \neq 0 \quad \text{for some } i$$

$$H_0: \beta_j = 0 \quad \text{for all } j$$
$$H_1: \beta_j \neq 0 \quad \text{for some } j$$

$$H_0: \alpha\beta_{ij} = 0 \quad \text{for all } ij$$
$$H_1: \alpha\beta_{ij} \neq 0 \quad \text{for some } ij.$$

The level of significance adopted for all tests is .05.

A total of eight subjects representing two random samples of four subjects each has been obtained from a common population. The two samples of subjects are randomly assigned to the $p = 2$ levels of A and observed under all $q = 4$ levels of B. The dependent variable is designated as response latency to the auditory and visual signals. Response is appropriate for many research situations. For example, in the vigilance experiment, subjects assigned to the auditory display condition a_1 might find this mode of signal presentation unpleasant. As a result, two of the subjects might refuse to complete the experiment. In this example, unequal cell frequencies result from the nature of the experimental treatments. A least-squares analysis, rather than an unweighted-means analysis, should be used. Computational formulas based on the data in Table 8.10-1 are given in Table 8.10-3.

TABLE 8.10-3 Computational Procedures for Least-Squares Solution for Type SPF-2.4 Design

$$SS_{\text{total}} = [ABS] - [X] = 235.500$$
$$SS_{\text{between subj}} = [AS] - [X] = 12.500$$
$$SS_A = [A] - [X] = 5.633$$
$$SS_{\text{subj w.groups}} = [AS] - [A] = 6.867$$
$$SS_{\text{within subj}} = [ABS] - [AS] = 223.000$$
$$SS_B = [B] - [X] = 194.500$$
$$SS_{AB} = [AB] - [A] - [B] + [X] = 15.634$$
$$SS_{B\times \text{subj w.groups}} = [ABS] - [AB] - [AS] + [A] = 12.866$$

The analysis is summarized in Table 8.10-4. In a least-squares analysis, the partitioned sum of squares add up to the total sum of squares.

Tests of simple main effects and comparisons among means have the same general form as tests based on equal cell frequencies. These procedures are illustrated in Sections 8.6 and 8.7. They generalize to the least-squares solution but require the substitution of the appropriate value for n.

TABLE 8.10-4 Analysis of Variance Table for Least-Squares Solution

Source	SS	df	MS	F
1 Between subjects	12.500	$N - 1 = 7$		
2 A	5.633	$p - 1 = 1$	5.633	$[\frac{2}{3}] = 4.92$
3 Subj w.groups	6.867	$N - p = 6$	1.144	
4 Within subjects	223.000	$N(q - 1) = 24$		
5 B	194.500	$q - 1 = 3$	64.833	$[\frac{5}{7}] = 90.68$
6 AB	15.634	$(p - 1)(q - 1) = 3$	5.211	$[\frac{6}{7}] = 7.29$
7 $B \times$ subj w.groups	12.866	$(N - p)(q - 1) = 18$	.715	
8 Total	235.500	$Nq - 1 = 31$		

*$p < .01$.

The two solutions for unequal n's illı strated in this section eac'ı require that the number of observations with n each block be equal.

THE PROBLEM OF ONE MISSING SCORE

The preceding solutions are appropri ıte when an entire bloc is missing. The following procedure is applicab e when only one score a block is missing. It is analogous to the metho in Section 5.6 for estima ıg missing values in a randomized block design. A missing score is estimate ıy

$$ABS_{ijm} = \frac{n(\Sigma S_m) + q(\Sigma AB_{ij}) - \Sigma A_i}{(n-1)(q-1)},$$

where = number of blocks in level A_i.

q = number of levels of B.

ΣS_m = sum of remaining scores in block containing missing score.

ΣAB_{ij} = sum of remaining scores in treatment combination AB_{ij} c ı-taining missing score.

ΣA_i = sum of remaining scores in treatment A_i containing mis ıg score.

For ,ample, assume that score ABS_{122} in Table 8.2-1 is missing. is scoı s estimated by

$$ABS_{122} = \frac{4(22) + 4(11) - 86}{(4-1)(4-1)} = 5.1,$$

whc $\Sigma S_2 = 27 - 5 = 22.$

$\Sigma AB_{12} = 16 - 5 =$.

$\Sigma A_1 = 91 - 5 =$).

Th stimated score is r ısonably close to the original score in that ll, wh is 5.

After inserting the stimate of the missing score into the data ma ıx, the nalysis of variance i carried out in the normal way. The degree of fre ım for $MS_{B \times \text{subj w. groups}}$ should be reduced by one; for exam le, df $p(n-1)(q-1) - 1$. An unbiased estimate of $MS_{B \times \text{subj w.groups}}$ is ob ned by this procedı e, but all other mean squares are slightly ov r-est nated. According to Anderson (1946), the biases are small. He gi es mı hods for obtaining uı biased estimates, but it is doubtful if the add d lal ır is justified. If anoth r missing score occurs in the same A_i treatme ıt, th iterative procedure described in Chapter 5 may be used. If the seco d m sing score occurs in a different level of treatment A, the procedure of es mating the score described above is repeated. A more complete d s-cu sion of procedures for estimating missing scores may be found in And r-so (1946) and Khargonkar (1948).

Standard error formulas due to Anderson (1946) for making comparisons among means by the t test, when only one score has been estimated, are shown below.

Comparisons among $\bar{A}_i$ means can be made using the following t denominator:

$$\sqrt{\frac{2[\text{MS}_{\text{subj w.groups}} + \frac{1}{2}(n-1)(q-1)(\text{MS}_{B\times\text{subj w.groups}})]}{nq}}.$$

Comparisons among $\bar{B}_j$ means employ the following t denominator:

$$\sqrt{\frac{2\text{MS}_{B\times\text{subj w.groups}}[1 + \frac{1}{2}(n-1)(q-1)(q/p)]}{np}}.$$

Comparisons among $\overline{AB}_{ij}$ means at level a_i use the following denominator:

$$\sqrt{\frac{2\text{MS}_{B\times\text{subj w.groups}}[1 + \frac{1}{2}(n-1)(q-1)(q/p)]}{n}}.$$

Comparisons among $\overline{AB}_{ij}$ means at level b_j use the following denominator:

$$\sqrt{\frac{2\text{MS}_{\text{subj w.groups}}/nq + 2\text{MS}_{B\times\text{subj w.groups}}[(q-1) + \frac{1}{2}(n-1)(q-1)(q^2)]}{nq}}.$$

The foregoing formulas are used in comparing means based on one estimated missing score. Procedures for determining the critical value for the t test for pooled error terms appear in Section 8.7. If a comparison among means does not involve a missing score, formulas given in Section 8.7 are appropriate.

If several missing scores occur in designs having three or more treatments, the reader should consult Hazel (1946), Henderson (1953), and Krishna Iyer (1940).

8.11 RELATIVE EFFICIENCY OF SPLIT-PLOT DESIGN

An experimenter wishing to use a multitreatment factorial design with subjects assigned to blocks may consider two of the designs described thus far—a randomized block factorial design and a split-plot design. However, he should examine several factors in choosing between these two designs. If it is not possible to administer all treatment level combinations within each block, there is no choice. A split-plot design is required. On the other hand, if there is a choice concerning the assignment of treatment combinations in each block, the relative efficiency of the A, *B*, and *AB* comparisons should be considered. In a split-plot design, the *B* and *AB* effects are usually estimated more accurately than the *A* effects. This results from the fact that variation within a block is usually smaller than variation among blocks. The average standard error of a difference is equal for both the randomized block factorial design and the split-plot design. Thus the increased accuracy of the *B* and *AB* effects estimates is obtained by sacrificing accuracy on the *A* effects. If the experimenter is as interested in the *A* effects as he is in the *B* and *AB* effects, the randomized block factorial design should be used. It should also be noted that the *F* ratio denominator degrees of freedom for *A*, *B*, and *AB* in the randomized block factorial design are larger than the corresponding degrees of freedom in a split-plot design.

A numerical index of relative efficiency of the two designs, disregarding differences in degrees of freedom, is given by the following formulas (Federer, 1955, 274). The data used in this example are from Table 8.2-2.

A efficiency

$$= \frac{[(p-1)\text{MS}_{\text{subj w.groups}} + p(n-1)\text{MS}_{B\times\text{subj w.groups}}]/(pq-1)}{\text{MS}_{\text{subj w.groups}}}$$

$$= \frac{[(2-1)1.562 + 2(4-1).507]/[(2)(4)-1]}{1.563} = \frac{.658}{1.563} \times 100$$

$$= 42.1 \text{ percent.}$$

B and *AB* efficiency

$$= \frac{[(p-1)\text{MS}_{\text{subj w.groups}} + p(n-1)\text{MS}_{B\times\text{subj w.groups}}]/(pq-1)}{\text{MS}_{B\times\text{subj w.groups}}}$$

$$= \frac{[(2-1)1.563 + 2(4-1).507]/[(2)(4)-1]}{.507} = \frac{.658}{.507} \times 100$$

$$= 129.8 \text{ percent.}$$

Hence, in this example, a test of the *A* treatment is less than half as efficient in the split-plot design as it is in the randomized block factorial design. On the other hand, the *B* and *AB* tests are more efficient in the split-plot design. The relative efficiency of tests is a basic consideration in the design of experiments.

8.12 INTRODUCTION TO TYPE SPF-*pr.q* DESIGN

The split-plot design described so far in this chapter has had two treatments. The general analysis procedure for a two-treatment split-plot design can be extended to designs having three or more treatments.

TABLE 8.13-2 Analysis of Variance Table for Type SPF-*pru.q* Design

Source	*Regular* df	*Unweighted-means* df	*F* (*A*, *B*, *C*, *and D Fixed Effects, Subjects Random*)
1 Between subjects	$npru - 1$	$N - 1$	
2 *A*	$p - 1$	$p - 1$	$[\frac{2}{9}]$
3 *C*	$r - 1$	$r - 1$	$[\frac{3}{9}]$
4 *D*	$u - 1$	$u - 1$	$[\frac{4}{9}]$
5 *AC*	$(p - 1)(r - 1)$	$(p - 1)(r - 1)$	$[\frac{5}{9}]$
6 *AD*	$(p - 1)(u - 1)$	$(p - 1)(u - 1)$	$[\frac{6}{9}]$
7 *CD*	$(r - 1)(u - 1)$	$(r - 1)(u - 1)$	$[\frac{7}{9}]$
8 *ACD*	$(p - 1)(r - 1)(u - 1)$	$(p - 1)(r - 1)(u - 1)$	$[\frac{8}{9}]$
9 Subj w.groups	$pru(n - 1)$	$N - pru$	
10 Within subjects	$npru(q - 1)$	$N(q - 1)$	
11 *B*	$q - 1$	$q - 1$	$[\frac{11}{19}]$
12 *AB*	$(p - 1)(q - 1)$	$(p - 1)(q - 1)$	$[\frac{12}{19}]$
13 *BC*	$(q - 1)(r - 1)$	$(q - 1)(r - 1)$	$[\frac{13}{19}]$
14 *BD*	$(q - 1)(u - 1)$	$(q - 1)(u - 1)$	$[\frac{14}{19}]$
15 *ABC*	$(p - 1)(q - 1)(r - 1)$	$(p - 1)(q - 1)(r - 1)$	$[\frac{15}{19}]$
16 *ABD*	$(p - 1)(q - 1)(u - 1)$	$(p - 1)(q - 1)(u - 1)$	$[\frac{16}{19}]$
17 *BCD*	$(q - 1)(r - 1)(u - 1)$	$(q - 1)(r - 1)(u - 1)$	$[\frac{17}{19}]$
18 *ABCD*	$(p - 1)(q - 1)(r - 1)(u - 1)$	$(p - 1)(q - 1)(r - 1)(u - 1)$	$[\frac{18}{19}]$
19 *B* × subj w.groups	$pru(n - 1)(q - 1)$	$(N - pru)(q - 1)$	
20 Total	$npqru - 1$	$Nq - 1$	

8.14 COMPUTATIONAL PROCEDURES FOR TYPE SPF-*p.qr* DESIGN

A split-plot design can be used in research situations requiring repeated measures on two or more treatments. One variation of this design, called a type SPF-*p.qr* design, is described here. This design has one between-block treatment (A) and two within-block treatments (B, C). The design requires np samples of subjects who are randomly assigned to treatment A, with n subjects (blocks) in each level. The sequence of administration of the BC treatment combinations within an np block is randomized independently for each subject. An alternative to using repeated measures on each subject is to randomly assign the BC treatment combinations to qr matched subjects within each block. This latter design requires a total of $npqr$ subjects.

These three-treatment designs are similar to split-split-plot designs used in agricultural research. The essential difference is that, in a split-split-plot design, levels of treatment A are assigned to plots (plots correspond to blocks of subjects in behavioral research). These plots are then subdivided for the levels of treatment B and subdivided again for the levels of treatment C. The levels of treatment B are randomly assigned to split-plots and levels of C are randomly assigned to split-split-plots. This randomization procedure can be contrasted with that used for a type SPF-$p.qr$ design, where the BC treatment combinations are randomly assigned within each block. The two randomization procedures lead to different error terms for testing treatment C and all interactions involving C.

Figure 8.14-1 shows a block diagram of a type SPF-2.22 design. The structural model for this design is

$$X_{ijkm} = \mu + \alpha_i + \pi_{m(i)} + \beta_j + \gamma_k + \alpha\beta_{ij} + \alpha\gamma_{ik} + \beta\gamma_{jk} + \alpha\beta\gamma_{ijk} + \beta\pi_{jm(i)} + \gamma\pi_{km(i)} + \beta\gamma\pi_{jkm(i)} + \varepsilon_{o(ijkm)}.$$

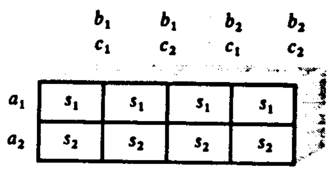


Figure 8.14-1 Block diagram of type SPF-2.22 design.

Because the analysis of this design poses computational problems not previously discussed, a numerical example is given in Table 8.14-1, where the following notation is used:

p levels of $a_i = 2$

q levels of $b_j = 2$

r levels of $c_k = 2$

n levels of $s_m = 4$

$$\sum_{1}^{N} = \sum_{1}^{p}\sum_{1}^{n}.$$

The analysis is summarized in Table 8.14-2.

TABLE 8.14-1 Computational Procedures for Type SPF-2.22 Design

(i) Data:

ABCS Summary Table

		$b_1 c_1$	$b_1 c_2$	$b_2 c_1$	$b_2 c_2$	$\sum_1^q \sum_1^r AS$	$\frac{\left(\sum_1^q \sum_1^r AS\right)^2}{qr}$
a_1	s_1	3	4	7	7	21	110.25
	s_2	6	5	8	8	27	182.25
	s_3	3	4	7	9	23	132.25
	s_4	3	3	6	8	20	100.00
a_2	s_5	1	2	5	10	18	81.00
	s_6	2	3	6	10	21	110.25
	s_7	2	4	5	9	20	100.00
	s_8	2	3	6	11	22	121.00

ABC Summary Table

	$b_1 c_1$	$b_1 c_2$	$b_2 c_1$	$b_2 c_2$
	$n = 4$			
a_1	15	16	28	32
a_2	7	12	22	40

AB Summary Table

	b_1	b_2	$\sum_1^q A$	$\frac{\left(\sum_1^q A\right)^2}{nqr}$
	$nr = 8$			
a_1	31	60	91	517.5625
a_2	19	62	81	410.0625

$\sum_1^p B = 50 \quad 122$

$\frac{\left(\sum_1^p B\right)^2}{npr} = 156.25 \quad 930.25$

AC Summary Table

	c_1	c_2
	$nq = 8$	
a_1	43	48
a_2	29	52

$\sum_1^p C = 72 \quad 100$

$\frac{\left(\sum_1^p C\right)^2}{npq} = 324 \quad 625$

BC Summary Table

	c_1	c_2
	$np = 8$	
b_1	22	28
b_2	50	72

ABS Summary Table

		b_1	b_2
		$r = 2$	
a_1	s_1	7	14
	s_2	11	16
	s_3	7	16
	s_4	6	14
a_2	s_5	3	15
	s_6	5	16
	s_7	6	14
	s_8	5	17

ACS Summary Table

		c_1	c_2
		$q = 2$	
a_1	s_1	10	11
	s_2	14	13
	s_3	10	13
	s_4	9	11
a_2	s_5	6	12
	s_6	8	13
	s_7	7	13
	s_8	8	14

TABLE 8.14-1 (continued)

(ii) Computational symbols:

$$\sum_1^q\sum_1^r\sum_1^N ABCS = 3 + 6 + 3 + \cdots + 11 = 172.000$$

$$\sum_1^q\sum_1^r\sum_1^N (ABCS)^2 = [ABCS] = (3)^2 + (6)^2 + \cdots + (11)^2 = 1160.000$$

$$\frac{\left(\sum_1^q\sum_1^r\sum_1^N ABCS\right)^2}{qrN} = [X] = \frac{(172)^2}{(2)(2)(8)} = 924.500$$

$$\sum_1^N \frac{\left(\sum_1^q\sum_1^r AS\right)^2}{qr} = [AS] = 110.25 + 182.25 + \cdots + 121.00 = 937.000$$

$$\sum_1^p \frac{\left(\sum_1^q A\right)^2}{nqr} = [A] = 517.5625 + 410.0625 = 927.625$$

$$\sum_1^q \frac{\left(\sum_1^p B\right)^2}{npr} = [B] = 156.25 + 930.25 = 1086.500$$

$$\sum_1^p\sum_1^q \frac{(AB)^2}{nr} = [AB] = \frac{(31)^2}{8} + \frac{(60)^2}{8} + \cdots + \frac{(62)^2}{8} = 1095.750$$

$$\sum_1^p\sum_1^q\sum_1^n \frac{(ABS)^2}{r} = [ABS] = \frac{(7)^2}{2} + \frac{(14)^2}{2} + \cdots + \frac{(17)^2}{2} = 1110.000$$

$$\sum_1^r \frac{\left(\sum_1^p C\right)^2}{npq} = [C] = 324 + 625 = 949.000$$

$$\sum_1^p\sum_1^r \frac{(AC)^2}{nq} = [AC] = \frac{(43)^2}{8} + \frac{(48)^2}{8} + \cdots + \frac{(52)^2}{8} = 962.250$$

$$\sum_1^p\sum_1^r\sum_1^n \frac{(ACS)^2}{q} = [ACS] = \frac{(10)^2}{2} + \frac{(11)^2}{2} + \cdots + \frac{(14)^2}{2} = 974.000$$

$$\sum_1^q\sum_1^r \frac{(BC)^2}{np} = [BC] = \frac{(22)^2}{8} + \frac{(28)^2}{8} + \cdots + \frac{(72)^2}{8} = 1119.000$$

$$\sum_1^p\sum_1^q\sum_1^r \frac{(ABC)^2}{n} = [ABC] = \frac{(15)^2}{4} + \frac{(16)^2}{4} + \cdots + \frac{(40)^2}{4} = 1141.500$$

(iii) Computational formulas:

$$SS_{\text{total}} = [ABCS] - [X] = 235.500$$

$$SS_{\text{between subj}} = [AS] - [X] = 12.500$$

$$SS_A = [A] - [X] = 3.125$$

$$SS_{\text{subj w.groups}} = [AS] - [A] = 9.375$$

$$SS_{\text{within subj}} = [ABCS] - [AS] = 223.000$$

$$SS_B = [B] - [X] = 162.000$$

$$SS_{AB} = [AB] - [A] - [B] + [X] = 6.125$$

$$SS_{B\times \text{subj w.groups}} = [ABS] - [AB] - [AS] + [A] = 4.875$$

TABLE 8.14-1 (continued)

$$SS_C = [C] - [X] = 24.500$$
$$SS_{AC} = [AC] - [A] - [C] + [X] = 10.125$$
$$SS_{C \times \text{subj w.groups}} = [ACS] - [AC] - [AS] + [A] = 2.375$$
$$SS_{BC} = [BC] - [B] - [C] + [X] = 8.000$$
$$SS_{ABC} = [ABC] - [AB] - [AC] - [BC] + [A] + [B] + [C] - [X] = 3.125$$
$$SS_{BC \times \text{subj w.groups}} = [ABCS] - [ABC] - [ABS] - [ACS] + [AB] + [AC] + [AS] - [A] = 1.875$$

TABLE 8.14-2 Analysis of Variance Table

	Source	SS	df
1	Between subjects	12.500	$np - 1 = 7$
2	*A*	3.125	$p - 1 = 1$
3	Subj w.groups	9.375	$p(n - 1) = 6$
4	Within subjects	223.000	$np(qr - 1) = 24$
5	*B*	162.000	$q - 1 = 1$
6	*AB*	6.125	$(p - 1)(q - 1) = 1$
7	*B* × subj w.groups	4.875	$p(n - 1)(q - 1) = 6$
8	*C*	24.500	$r - 1 = 1$
9	*AC*	10.125	$(p - 1)(r - 1) = 1$
10	*C* × subj w.groups	2.375	$p(n - 1)(r - 1) = 6$
11	*BC*	8.000	$(q - 1)(r - 1) = 1$
12	*ABC*	3.125	$(p - 1)(q - 1)(r - 1) = 1$
13	*BC* × subj w.groups	1.875	$p(n - 1)(q - 1)(r - 1) = 6$
14	Total	235.500	$npqr - 1 = 31$

*$p < .05$.
**$p < .01$.

TESTS FOR HOMOGENEITY OF ERROR TERMS

Four sets of error terms in a type SPF-2.22 design can be tested for homogeneity. The variances estimated by $MS_{\text{subj w.group } a_i}$ at p levels of A should be homogeneous. Similarly, the variances estimated by $MS_{B \times \text{subj w.group } a_i}$ at p levels of A should be homogeneous, and the same is true for $MS_{C \times \text{subj w.group } a_i}$ and $MS_{BC \times \text{subj w.group } a_i}$. Computational procedures for computing the required mean squares at level a_1 appear in Table 8.14-3. The formulas at level a_2 follow the same pattern as those at level a_1. An F_{max} ratio for these partitioned error terms has the form

$$F_{\max} = \left[\frac{\text{largest MS}}{\text{smallest MS}}\right] \text{with } \nu_1 \text{ and } \nu_2 \text{ degrees of freedom.}$$

Six variance-covariance matrices are associated with a type SPF-2.22 design—$q \times q$ at a_1, $q \times q$ at a_2, $r \times r$ at a_1, $r \times r$ at a_2, $qr \times qr$ at a_1, and $qr \times qr$ at a_2. In order for each of the within-subjects F ratios to be distributed as the F distribution, all variances for a particular population dispersion matrix should be equal to σ^2 and all covariances equal to $\rho\sigma^2$ Section 8.5 described procedures for determining the tenability of the hypotheses that, for example, (1) the $q \times q$ population matrix at level a_1 is equal to the $q \times q$ matrix at a_2 and (2) the pooled dispersion matrix

MS	F	E(MS) *A, B, and C Fixed Effects, Subjects Random*
3.125	$[\frac{2}{3}] =$ 2.00	$\sigma_\varepsilon^2 + qr\sigma_\pi^2 + nqr\sigma_\alpha^2$
1.562		$\sigma_\varepsilon^2 + qr\sigma_\pi^2$
162.000	$[\frac{5}{7}] =$ 199.51**	$\sigma_\varepsilon^2 + r\sigma_{\beta\pi}^2 + npr\sigma_\beta^2$
6.125	$[\frac{6}{7}] =$ 7.54*	$\sigma_\varepsilon^2 + r\sigma_{\beta\pi}^2 + nr\sigma_{\alpha\beta}^2$
.812		$\sigma_\varepsilon^2 + r\sigma_{\beta\pi}^2$
24.500	$[\frac{8}{10}] =$ 61.87**	$\sigma_\varepsilon^2 + q\sigma_{\gamma\pi}^2 + npq\sigma_\gamma^2$
10.125	$[\frac{9}{10}] =$ 25.57**	$\sigma_\varepsilon^2 + q\sigma_{\gamma\pi}^2 + nq\sigma_{\alpha\gamma}^2$
.396		$\sigma_\varepsilon^2 + q\sigma_{\gamma\pi}^2$
8.000	$[\frac{11}{13}] =$ 25.64**	$\sigma_\varepsilon^2 + \sigma_{\beta\gamma\pi}^2 + np\sigma_{\beta\gamma}^2$
3.125	$[\frac{12}{13}] =$ 10.02*	$\sigma_\varepsilon^2 + \sigma_{\beta\gamma\pi}^2 + n\sigma_{\alpha\beta\gamma}^2$
.312		$\sigma_\varepsilon^2 + \sigma_{\beta\gamma\pi}^2$

for the two levels of A has the symmetry described above. Identical tests can be performed for the $r \times r$ dispersion matrices and $qr \times qr$ dispersion matrices. If the assumptions of equality and symmetry of the variance-covariance matrices are not tenable, conservative F tests as presented in Table 8.14-4 can be computed. It will be recalled from Section 8.5 that a conservative F test is computed in the usual way but that the F table is entered with modified degrees of freedom.

TESTS OF SIMPLE EFFECTS

It is apparent from Table 8.14-2 that treatments B and C, as well as the AB, AC, BC, and ABC interactions, are significant. Because the triple

TABLE 8.14-3 Partitioning of Error Terms for Tests of Homogeneity

$$SS_{\text{subj w.group } a_1} = \sum_1^n \frac{\left(\sum_1^q\sum_1^r AS_{1jkm}\right)^2}{qr} - \frac{\left(\sum_1^q A_{1j}\right)^2}{nqr}$$

$$df = n - 1$$

$$SS_{B\times\text{subj w.group } a_1} = \sum_1^q\sum_1^n \frac{(ABS_{1jm})^2}{r} - \sum_1^q \frac{(AB_{1j})^2}{nr} - \sum_1^n \frac{\left(\sum_1^q\sum_1^r AS_{1jkm}\right)^2}{qr} + \frac{\left(\sum_1^q A_{1j}\right)^2}{nqr}$$

$$df = (n-1)(q-1)$$

$$SS_{C\times\text{subj w.group } a_1} = \sum_1^r\sum_1^n \frac{(ACS_{1km})^2}{q} - \sum_1^r \frac{(AC_{1k})^2}{nq} - \sum_1^n \frac{\left(\sum_1^q\sum_1^r AS_{1jkm}\right)^2}{qr} + \frac{\left(\sum_1^q A_{1j}\right)^2}{nqr}$$

$$df = (n-1)(r-1)$$

$$SS_{BC\times\text{subj w.group } a_1} = \sum_1^q\sum_1^r\sum_1^n ABCS_{1jkm}^2 - \sum_1^q\sum_1^r \frac{(ABC_{1jk})^2}{n} - \sum_1^q\sum_1^n \frac{(ABS_{1jm})^2}{r} - \sum_1^r\sum_1^n \frac{(ACS_{1km})^2}{q}$$

$$+ \sum_1^q \frac{(AB_{1j})^2}{nr} + \sum_1^r \frac{(AC_{1k})^2}{nq} + \sum_1^n \frac{\left(\sum_1^q\sum_1^r AS_{1jkm}\right)^2}{qr} - \frac{\left(\sum_1^q A_{1j}\right)^2}{nqr}$$

$$df = (n-1)(q-1)(r-1)$$

TABLE 8.14-4 Conservative F Ratio Degrees of Freedom

Numerator of F ratio	df
MS_B	$1, p(n-1)$
MS_{AB}	$(p-1), p(n-1)$
MS_C	$1, p(n-1)$
MS_{AC}	$p-1, p(n-1)$
MS_{BC}	$1, p(n-1)$
MS_{ABC}	$p-1, p(n-1)$

interaction is significant, the experimenter might want to test the significance of simple-simple main effects and simple interaction effects. These tests are outlined in Table 8.14-5. The formulas are given for the first level of each treatment only. The formulas at other levels of the treatments follow the pattern given for the first level. The error terms appearing in Table 8.14-5 for each test are appropriate for a mixed model.

The rationale underlying the selection of a denominator for F ratios is presented in Section 8.6. The reader should be certain that he understands the principles on which the determination of the correct error terms is based. The terms given in the table are only correct for the mixed model in which A, B, and C are fixed effects and subjects are random effects. No difficulty should be encountered in determining the error term for other models if one remembers that

TABLE 8.14-5 Formulas for Testing Simple Effects

	Error Term (*A, B, and C Fixed Effects, Subjects Random*)
(i) *AB* interaction significant:	
$SS_A \text{ at } b_1 = \sum_1^p \frac{(AB_{i1})^2}{nr} - \frac{\left(\sum_1^p B_{i1}\right)^2}{npr}$	$\frac{MS_{\text{subj w.groups}} + MS_{B\times\text{subj w.groups}}(q-1)}{q}$
$SS_B \text{ at } a_1 = \sum_1^q \frac{(AB_{1j})^2}{nr} - \frac{\left(\sum_1^q A_{1j}\right)^2}{nqr}$	$MS_{B\times\text{subj w.groups}}$
(ii) *AC* interaction significant:	
$SS_A \text{ at } c_1 = \sum_1^p \frac{(AC_{i1})^2}{nq} - \frac{\left(\sum_1^p C_{i1}\right)^2}{npq}$	$\frac{MS_{\text{subj w.groups}} + MS_{C\times\text{subj w.groups}}(r-1)}{r}$
$SS_C \text{ at } a_1 = \sum_1^r \frac{(AC_{1k})^2}{nq} - \frac{\left(\sum_1^r A_{1k}\right)^2}{nqr}$	$MS_{C\times\text{subj w.groups}}$
(iii) *BC* interaction significant:	
$SS_B \text{ at } c_1 = \sum_1^q \frac{(BC_{j1})^2}{np} - \frac{\left(\sum_1^q C_{j1}\right)^2}{npq}$	$\frac{MS_{B\times\text{subj w.groups}} + MS_{BC\times\text{subj w.groups}}(r-1)}{r}$
$SS_C \text{ at } b_1 = \sum_1^r \frac{(BC_{1k})^2}{np} - \frac{\left(\sum_1^r B_{1k}\right)^2}{npr}$	$\frac{MS_{C\times\text{subj w.groups}} + MS_{BC\times\text{subj w.groups}}(q-1)}{q}$
(iv) *ABC* interaction significant:	
$SS_A \text{ at } bc_{11} = \sum_1^p \frac{(ABC_{i11})^2}{n} - \frac{(BC_{11})^2}{np}$	$MS_{\text{w.cell}}$
$SS_B \text{ at } ac_{11} = \sum_1^q \frac{(ABC_{1j1})^2}{n} - \frac{(AC_{11})^2}{nq}$	$\frac{MS_{B\times\text{subj w.groups}} + MS_{BC\times\text{subj w.groups}}(r-1)}{r}$
$SS_C \text{ at } ab_{11} = \sum_1^r \frac{(ABC_{11k})^2}{n} - \frac{(AB_{11})^2}{nr}$	$\frac{MS_{C\times\text{subj w.groups}} + MS_{BC\times\text{subj w.groups}}(q-1)}{q}$
$SS_{AB} \text{ at } c_1 = \left[\sum_1^p\sum_1^q \frac{(ABC_{ij1})^2}{n} - \frac{\left(\sum_1^p C_{i1}\right)^2}{npq}\right] - SS_A \text{ at } c_1 - SS_B \text{ at } c_1$	$\frac{MS_{B\times\text{subj w.groups}} + MS_{BC\times\text{subj w.groups}}(r-1)}{r}$
$SS_{AC} \text{ at } b_1 = \left[\sum_1^p\sum_1^r \frac{(ABC_{i1k})^2}{n} - \frac{\left(\sum_1^p B_{i1}\right)^2}{npr}\right] - SS_A \text{ at } b_1 - SS_C \text{ at } b_1$	$\frac{MS_{C\times\text{subj w.groups}} + MS_{BC\times\text{subj w.groups}}(q-1)}{q}$
$SS_{BC} \text{ at } a_1 = \left[\sum_1^q\sum_1^r \frac{(ABC_{1jk})^2}{n} - \frac{\left(\sum_1^q A_{1j}\right)^2}{nqr}\right] - SS_B \text{ at } a_1 - SS_C \text{ at } a_1$	$MS_{BC\times\text{subj w.groups}}$

$$\sum_{1}^{q} \mathrm{SS}_A \text{ for } b_j = \mathrm{SS}_A + \mathrm{SS}_{AB}$$

$$\sum_{1}^{q}\sum_{1}^{r} \mathrm{SS}_A \text{ for } bc_{jk} = \mathrm{SS}_A + \mathrm{SS}_{AB} + \mathrm{SS}_{AC} + \mathrm{SS}_{ABC}$$

$$\sum_{1}^{r} \mathrm{SS}_{AB} \text{ for } c_k = \mathrm{SS}_{AB} + \mathrm{SS}_{ABC}.$$

One can see that whenever the main effects have different error terms, these error terms are pooled in testing the simple main effects. The error terms in Table 8.14-5 are given as weighted pooled mean squares divided by their pooled degrees of freedom. For example, the error term for testing SS_A at b_1 can be computed by either of the following formulas:

$$\frac{\mathrm{MS}_{\text{subj w.groups}} + \mathrm{MS}_{B \times \text{subj w.groups}}(q - 1)}{q} = \frac{1.562 + .812(1)}{2} = 1.187$$

$$\frac{\mathrm{SS}_{\text{subj w.groups}} + \mathrm{SS}_{B \times \text{subj w.groups}}}{p(n - 1) + p(n - 1)(q - 1)} = \frac{9.375 + 4.875}{2(3) + 2(3)(1)} = 1.188.$$

The two answers agree within rounding error.

COMPARISONS AMONG MEANS

Tests of differences among means follow the procedures described in Section 8.7. For example, the error term for Tukey's ratio for the comparison $\hat{\psi} = \bar{A}_1 - \bar{A}_2$ is

$$\sqrt{\frac{\mathrm{MS}_{\text{subj w.groups}}}{nqr}}.$$

The divisor, nqr, is the sample size for $\bar{A}_i$. The comparison $\hat{\psi} = \overline{AB}_{11} - \overline{AB}_{21}$ has as its error term

$$\sqrt{\frac{\mathrm{MS}_{\text{subj w.groups}} + \mathrm{MS}_{B \times \text{subj w.groups}}(q - 1)}{nr(q)}}.$$

The term $[\mathrm{MS}_{\text{subj w.groups}} + \mathrm{MS}_{B \times \text{subj w.groups}}(q - 1)]/q$ is the F ratio denominator for testing MS_A at b_j. The term $nr = 8$ is the number of scores in each cell of the AB Summary Table.

EXPECTED VALUES OF MEAN SQUARES

The expected values of mean squares for Models I, II, and III can be determined from Table 8.14-6. The terms $1 - p/P$, $1 - q/Q$, $1 - r/R$, and $1 - n/N$ become zero if the corresponding effects are fixed and one if the effects are random.

TABLE 8.14-6 Table for Determining E(MS) for Type SPF-$p.qr$ Design

Source	E(MS)
A	$\sigma_\varepsilon^2 + \left(1-\frac{n}{N}\right)\left(1-\frac{q}{Q}\right)\left(1-\frac{r}{R}\right)\sigma_{\beta\gamma\pi}^2 + n\left(1-\frac{q}{Q}\right)\left(1-\frac{r}{R}\right)\sigma_{\alpha\beta\gamma}^2 + q\left(1-\frac{n}{N}\right)\left(1-\frac{r}{R}\right)\sigma_{\gamma\pi}^2 + nq\left(1-\frac{r}{R}\right)\sigma_{\alpha\gamma}^2 + r\left(1-\frac{n}{N}\right)\left(1-\frac{q}{Q}\right)\sigma_{\beta\pi}^2 + nr\left(1-\frac{q}{Q}\right)\sigma_{\alpha\beta}^2 + qr\left(1-\frac{n}{N}\right)\sigma_\pi^2 + nqr\sigma_\alpha^2$
Subj w.groups	$\sigma_\varepsilon^2 + \left(1-\frac{q}{Q}\right)\left(1-\frac{r}{R}\right)\sigma_{\beta\gamma\pi}^2 + q\left(1-\frac{r}{R}\right)\sigma_{\gamma\pi}^2 + r\left(1-\frac{q}{Q}\right)\sigma_{\beta\pi}^2 + qr\sigma_\pi^2$
B	$\sigma_\varepsilon^2 + \left(1-\frac{n}{N}\right)\left(1-\frac{r}{R}\right)\sigma_{\beta\gamma\pi}^2 + n\left(1-\frac{p}{P}\right)\left(1-\frac{r}{R}\right)\sigma_{\alpha\beta\gamma}^2 + np\left(1-\frac{r}{R}\right)\sigma_{\beta\gamma}^2 + r\left(1-\frac{n}{N}\right)\sigma_{\beta\pi}^2 + nr\left(1-\frac{p}{P}\right)\sigma_{\alpha\beta}^2 + npr\sigma_\beta^2$
AB	$\sigma_\varepsilon^2 + \left(1-\frac{n}{N}\right)\left(1-\frac{r}{R}\right)\sigma_{\beta\gamma\pi}^2 + n\left(1-\frac{r}{R}\right)\sigma_{\alpha\beta\gamma}^2 + r\left(1-\frac{n}{N}\right)\sigma_{\beta\pi}^2 + nr\sigma_{\alpha\beta}^2$
$B \times$ subj w.groups	$\sigma_\varepsilon^2 + \left(1-\frac{r}{R}\right)\sigma_{\beta\gamma\pi}^2 + r\sigma_{\beta\pi}^2$
C	$\sigma_\varepsilon^2 + \left(1-\frac{n}{N}\right)\left(1-\frac{q}{Q}\right)\sigma_{\beta\gamma\pi}^2 + n\left(1-\frac{p}{P}\right)\left(1-\frac{q}{Q}\right)\sigma_{\alpha\beta\gamma}^2 + np\left(1-\frac{q}{Q}\right)\sigma_{\beta\gamma}^2 + q\left(1-\frac{n}{N}\right)\sigma_{\gamma\pi}^2 + nq\left(1-\frac{p}{P}\right)\sigma_{\alpha\gamma}^2 + npq\sigma_\gamma^2$
AC	$\sigma_\varepsilon^2 + \left(1-\frac{n}{N}\right)\left(1-\frac{q}{Q}\right)\sigma_{\beta\gamma\pi}^2 + n\left(1-\frac{q}{Q}\right)\sigma_{\alpha\beta\gamma}^2 + q\left(1-\frac{n}{N}\right)\sigma_{\gamma\pi}^2 + nq\sigma_{\alpha\gamma}^2$
$C \times$ subj w.groups	$\sigma_\varepsilon^2 + \left(1-\frac{q}{Q}\right)\sigma_{\beta\gamma\pi}^2 + q\sigma_{\gamma\pi}^2$
BC	$\sigma_\varepsilon^2 + \left(1-\frac{n}{N}\right)\sigma_{\beta\gamma\pi}^2 + n\left(1-\frac{p}{P}\right)\sigma_{\alpha\beta\gamma}^2 + np\sigma_{\beta\gamma}^2$
ABC	$\sigma_\varepsilon^2 + \left(1-\frac{n}{N}\right)\sigma_{\beta\gamma\pi}^2 + n\sigma_{\alpha\beta\gamma}^2$
$BC \times$ subj w.groups	$\sigma_\varepsilon^2 + \sigma_{\beta\gamma\pi}^2$

TABLE 8.15-1 (continued)

	$+ [ABD] + [ACD]$	
	$+ [ABS] + [ACS]$	
	$+ [ADS] - [AB] - [AC]$	
	$- [AD] - [AS] + [A]$	$p(n - 1)(q - 1)(r - 1)(u - 1)$
26	$SS_{total} = [ABCDS] - [X]$	$npqru - 1$

8.16 COMPUTATIONAL PROCEDURES FOR TYPE SPF-*pr.qu* DESIGN

A type SPF-*pr.qu* design represents an extension of analysis procedures described for type *pr.q* and *p.qr* designs. A block diagram of this design appears in Figure 8.16-1 The structural model for the design is

$$X_{ijklm} = \mu + \alpha_i + \gamma_k + \alpha\gamma_{ik} + \pi_{m(ik)} + \beta_j + \alpha\beta_{ij} + \beta\gamma_{jk} + \alpha\beta\gamma_{ijk} + \beta\pi_{jm(ik)}$$
$$+ \delta_l + \alpha\delta_{il} + \gamma\delta_{kl} + \alpha\gamma\delta_{ikl} + \delta\pi_{lm(ik)} + \beta\delta_{jl} + \alpha\beta\delta_{ijl} + \beta\gamma\delta_{jkl}$$
$$+ \alpha\beta\gamma\delta_{ijkl} + \beta\delta\pi_{jlm(ik)} + \varepsilon_{o(ijklm)}$$

	b_1 d_1	b_1 d_2	b_2 d_1	b_2 d_2
ac_{11}	s_1	s_1	s_1	s_1
ac_{12}	s_2	s_2	s_2	s_2
ac_{21}	s_3	s_3	s_3	s_3
ac_{22}	s_4	s_4	s_4	s_4

Figure 8.16-1 Block diagram of type SPF-22.22 design

The computational formulas for the design, degrees of freedom, and *F* ratios for Model III appear in Table 8.16-1. The meaning of the terms should be clear from previous examples.

TABLE 8.16-1 Computational Formulas for Type SPF-*pr.qu* Design and *F* Ratios

	Computational Formulas	df	*F ratio (A, B, C and D Fixed Effects Subjects Random)*
1	$SS_{between\ subj} = [ACS] - [X]$	$npr - 1$	
2	$SS_A = [A] - [X]$	$p - 1$	$[\frac{2}{5}]$
3	$SS_C = [C] - [X]$	$r - 1$	$[\frac{3}{5}]$

TABLE 8.16-1 (continued)

4	$SS_{AC} = [AC] - [A] - [C] + [X]$	$(p-1)(r-1)$	$[\frac{4}{5}]$
5	$SS_{subj\ w.groups} = [ACS] - [AC]$	$pr(n-1)$	
6	$SS_{within\ subj} = [ABCDS] - [ACS]$	$npr(qu-1)$	
7	$SS_B = [B] - [X]$	$q-1$	$[\frac{7}{11}]$
8	$SS_{AB} = [AB] - [A] - [B] + [X]$	$(p-1)(q-1)$	$[\frac{8}{11}]$
9	$SS_{BC} = [BC] - [B] - [C] + [X]$	$(q-1)(r-1)$	$[\frac{9}{11}]$
10	$SS_{ABC} = [ABC] - [AB] - [AC]$ $- [BC] + [A] + [B]$ $+ [C] - [X]$	$(p-1)(q-1)(r-1)$	$[\frac{10}{11}]$
11	$SS_{B \times subj\ w.groups} = [ABCS] - [ABC] - [ACS]$ $+ [AC]$	$pr(n-1)(q-1)$	
12	$SS_D = [D] - [X]$	$u-1$	$[\frac{12}{16}]$
13	$SS_{AD} = [AD] - [A] - [D] + [X]$	$(p-1)(u-1)$	$[\frac{13}{16}]$
14	$SS_{CD} = [CD] - [C] - [D] + [X]$	$(r-1)(u-1)$	$[\frac{14}{16}]$
15	$SS_{ACD} = [ACD] - [AC] - [AD]$ $- [CD] + [A] + [C]$ $+ [D] - [X]$	$(p-1)(r-1)(u-1)$	$[\frac{15}{16}]$
16	$SS_{D \times subj\ w.groups} = [ACDS] - [ACD] - [ACS]$ $+ [AC]$	$pr(n-1)(u-1)$	
17	$SS_{BD} = [BD] - [B] - [D] + [X]$	$(q-1)(u-1)$	$[\frac{17}{21}]$
18	$SS_{ABD} = [ABD] - [AB] - [AD]$ $- [BD] + [A] + [B]$ $+ [D] - [X]$	$(p-1)(q-1)(u-1)$	$[\frac{18}{21}]$
19	$SS_{BCD} = [BCD] - [BC] - [BD]$ $- [CD] + [B] + [C]$ $+ [D] - [X]$	$(q-1)(r-1)(u-1)$	$[\frac{19}{21}]$
20	$SS_{ABCD} = [ABCD] - [ABC] - [ABD]$ $[ACD] - [BCD] + [AB]$ $+ [AC] + [AD] + [BC]$ $+ [BD] + [CD] - [A] - [B]$ $- [C] - [D] + [X]$	$(p-1)(q-1)(r-1)(u-1)$	$[\frac{20}{21}]$
21	$SS_{BD \times subj\ w.groups} = [ABCDS] - [ABCD]$ $- [ABCS] - [ACDS]$ $+ [ABC] + [ACD] + [ACS]$ $- [AC]$	$pr(n-1)(q-1)(u-1)$	
22	$SS_{total} = [ABCDS] - [X]$	$npqru - 1$	