2.5 THE *F* RATIO IN ANALYSIS OF VARIANCE

An F ratio in analysis of variance provides a test of the hypothesis that all treatment population means are equal. That is,

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k = \mu.$$

This null hypothesis is equivalent to the hypothesis that

$$H_0: \beta_j = 0 \qquad \text{for all } j.$$

We have seen that when H_0 is true,

$$\text{E}(\text{MS}_{\text{BG}}) = \sigma_\varepsilon^2$$

and

$$\text{E}(\text{MS}_{\text{WG}}) = \sigma_\varepsilon^2.$$

When the null hypothesis is false and the alternative hypothesis that

$$H_1: \beta_j \neq 0 \qquad \text{for some } j$$

is true,

$$\text{E}(\text{MS}_{\text{BG}}) > \text{E}(\text{MS}_{\text{WG}}).$$

If the null hypothesis is true, we know from Section 2.1 that the random variables $\text{MS}_{\text{BG}}/\sigma_\varepsilon^2$ and $\text{MS}_{\text{WG}}/\sigma_\varepsilon^2$ are both distributed as chi-square variables divided by their respective degrees of freedom. Thus, if the null hypothesis is true, if MS_{BG} and MS_{WG} are statistically independent, and if the population variances are homogeneous, the ratio

$$\frac{\text{MS}_{\text{BG}}}{\text{MS}_{\text{WG}}} = \frac{\sigma_\varepsilon^2 \chi^2_{(\nu_1)}/\nu_1}{\sigma_\varepsilon^2 \chi^2_{(\nu_2)}/\nu_2} = F_{(\nu_1, \nu_2)}$$

is distributed as the F distribution, with $\nu_1 = k - 1$ and $\nu_2 = kn - k$ degrees of freedom. It can be stated without proof that the mean $\bar{X}_j$ and the variance $\hat{\sigma}_j^2$ estimates are statistically independent provided that the population is normally distributed. Hence MS_{BG} and MS_{WG} are independent as long as the k samples of observations are independently drawn from normally distributed populations. The probability of obtaining an F as large as that observed in an experiment if the null hypothesis is true can be determined from a table of F given in Appendix D.

An F ratio, as defined above, always provides a one-tailed test of H_0. Ratios less than 1.0 have no meaning with respect to H_0. Such ratios may occur as a result of the operation of chance, for both numerator and denominator are subject to sampling error. Ratios less than 1.0 may also occur because of failure to randomize some important factor properly in the experimental design or because some of the assumptions concerning the linear model for the design are inappropriate. In summary, if two sets of assumptions are tenable—those associated with the derivation of the